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Solving parabolic and hyperbolic equations by the generalized finite difference method

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Abstract

Classical finite difference schemes are in wide use today for approximately solving partial differential equations of mathematical physics. An evolution of the method of finite differences has been the development of generalized finite difference (GFD) method, that can be applied to irregular grids of points.

In this paper the extension of the GFD to the explicit solution of parabolic and hyperbolic equations has been developed for partial differential equations with constant coefficients in the cases of considering one, two or three space dimensions. The convergence of the method has been studied and the truncation errors over irregular grids are given.

Different examples have been solved using the explicit finite difference formulae and the criterion of stability. This has been expressed in function of the coefficients of the star equation for irregular clouds of nodes in one, two or three space dimensions. The numerical results show the accuracy obtained over irregular grids. This paper also includes the study of the maximum local error and the global error for different examples of parabolic and hyperbolic time-dependent equations.

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1. Introduction

Partial differential equations are the chief means of providing mathematical models in science, engineering and other fields. Generally these models must be solved numerically. In many practical problems the finite difference method is used.

An evolution of the method of finite differences has been the development of generalized finite difference (GFD) method, that can be applied to irregular grids or clouds of points. Lizska and Orkisz [5,6] proposed a generalized finite difference method (GFDM) on irregular grids. Their solution was obtained using moving least squares (MLS) approximation. However, these GFD formulations were later essentially improved and extended by many other authors and the most advanced version was given by Orkisz [8–10], including: mesh generation, local approximation, generation

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of finite difference (FD) formulae and FD equations resulting from local (collocation) or global (Galerkin, variational, ...) formulations.

Benito, Ureña and Gavete have made interesting contributions to the development of this method. The paper [1] shows explicit formulae for GFDM using irregular grids and the influence of parameters that define them. In the paper [4] a procedure is given that can easily assure the quality of numerical results by obtaining the residual at each point. Also, in [4], the GFDM is compared with another meshless method the, so-called, element free Galerkin method (EFG). The possibility of employing the GFD method over adaptive clouds of points progressively increasing the number of nodes is studied in [2].

The extension of the GFDM to the explicit solution of parabolic and hyperbolic equations is given in this paper. The convergence of the method is demonstrated and the stability criteria for irregular grids are given. Different examples are solved using the explicit finite difference formulae and the criterion of stability, in one, two or three space dimensions.

The method developed in this paper uses weighting functions, but it is not related with the well-known method of solving partial differential equations by collocation with radial basis functions of other authors, as in Fasshauer [3].

The paper is organized as follows. Section 2 describes the equations to be solved and the approximations of the space derivatives in 1D, 2D and 3D cases. Explicit finite difference schemes are considered in Section 3, and the study of convergence is given in Section 4. Numerical results for parabolic and hyperbolic equations are given in Section 5, and lastly, Section 6 gives the conclusions. Approximations of space derivatives for 1D case are given in Appendices A and B, and the truncation errors for 2D and 3D cases are given in Appendix C.

2. Finite difference formulae

In this work we consider the finite difference method for the solution of parabolic and hyperbolic partial differential equations with constant coefficients:

- The parabolic equations are defined by

$$\frac{\partial U(\{x\}, t)}{\partial t} = \Psi[U(\{x\}, t)], \quad t > 0, \quad \{x\} \in \Omega \quad (\Omega \subset R^1, R^2 \text{ or } R^3) \quad (1a)$$

with the initial condition

$$U(\{x\}, 0) = f(\{x\}) \quad (1b)$$

and the boundary condition

$$\alpha \frac{\partial U}{\partial n} + \beta U = g(t) \quad \text{in } \Gamma \quad (1c)$$

where $f(\{x\})$, $g(t)$ being two known functions, α , β are constants and Γ is the boundary of Ω .

- The hyperbolic equations are defined by

$$\frac{\partial^2 U(\{x\}, t)}{\partial t^2} = c^2 \Psi[U(\{x\}, t)] \quad t > 0, \quad \{x\} \in \Omega \quad (\Omega \subset R^1, R^2 \text{ or } R^3) \quad (2a)$$

with the initial conditions

$$\begin{cases} U(\{x\}, 0) = f(\{x\}) \\ \frac{\partial U(\{x\}, 0)}{\partial t} = h(\{x\}) \end{cases} \quad t > 0 \quad (2b)$$

and the boundary condition

$$\alpha \frac{\partial U}{\partial n} + \beta U = g(t) \quad \text{in } \Gamma, \quad (2c)$$

where c is constant, $f(\{x\})$, $g(t)$, $h(\{x\})$ are three known functions, α , β are constants and Γ is the boundary of Ω .

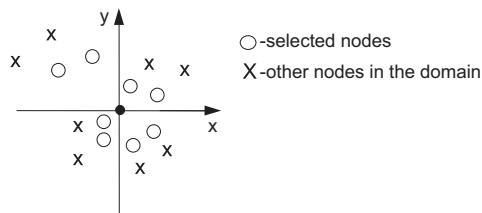


Fig. 1. The four quadrants criterium, (choices) two nodes in each quadrant.

In these equations (1a)–(1c), (2a)–(2c), $\{x\}$ refers to one, two or three space dimensions vector and Ψ is linear partial differential second order operator with constant coefficients in the space variables.

In order to obtain a GFD approximation of the partial differential equations, the domain Ω is covered by an irregular distribution of points. The aim is to establish a recursive relationship using an explicit formula for the part that depends on the time and the GFD for the space domain. Afterwards, these formulas are shown.

2.1. Approximation of the space variables by GFDs

Firstly, we consider the GFD method to solve the equation in the space variables

$$\Psi[U] = 0 \quad \text{in } \Omega \quad (3a)$$

with the boundary conditions

$$\alpha \frac{\partial U}{\partial n} + \beta U = g \quad \text{in } \Gamma, \quad (3b)$$

where $\Omega \subset R$ or $\Omega \subset R^2$ or $\Omega \subset R^3$ with boundary Γ , Ψ is a linear partial differential second order operator with constant coefficients, α, β are constants, g is a known function.

The intention is to obtain explicit linear expressions for the approximation of partial derivatives in the points of the domain. First of all, an irregular grid or cloud of points is generated in the domain $\Omega \cup \Gamma$. On defining the central node with a set of nodes surrounding that node, the star then refers to a group of established nodes in relation to a central node. Each node in the domain has an associated star assigned to it.

The choice of these supporting nodes of the star is constrained as particular patterns can lead to degenerated solutions. As star selection criterium we follow the denominated cross criterium: for example, in 2D case the area around the central nodal point 0 (in black in Fig. 1), is divided into four sectors corresponding to quadrants of the cartesian coordinates system originating at the central node (see Fig. 1). In each sector two or more nodes are selected, the closest to the origin (circles shown in Fig. 1). If this is not possible, e.g., at the boundary, missing nodes can be supplemented to provide the total number of nodes necessary in each star. Similarly, in 3D case the volume around the central node 0 is divided in eight sectors corresponding to octants of the Cartesian coordinates system originating at the central node.

If U_0 is the value of the function at the central node of the star and U_i are the function values at the rest of nodes, with $i = 1, \dots, N$, then, according to the Taylor series expansion in 1D, 2D and 3D, respectively,

$$U_i = U_0 + h_i \frac{\partial U_0}{\partial x} + \frac{1}{2} \left(h_i^2 \frac{\partial^2 U_0}{\partial x^2} \right) + \dots, \quad (4)$$

$$U_i = U_0 + h_i \frac{\partial U_0}{\partial x} + k_i \frac{\partial U_0}{\partial y} + \frac{1}{2} \left(h_i^2 \frac{\partial^2 U_0}{\partial x^2} + k_i^2 \frac{\partial^2 U_0}{\partial y^2} + 2h_i k_i \frac{\partial^2 U_0}{\partial x \partial y} \right) + \dots, \quad (5)$$

$$U_i = U_0 + h_i \frac{\partial U_0}{\partial x} + k_i \frac{\partial U_0}{\partial y} + p_i \frac{\partial U_0}{\partial z} + \frac{1}{2} \left(h_i^2 \frac{\partial^2 U_0}{\partial x^2} + k_i^2 \frac{\partial^2 U_0}{\partial y^2} + p_i^2 \frac{\partial^2 U_0}{\partial z^2} + 2h_i k_i \frac{\partial^2 U_0}{\partial x \partial y} + 2h_i p_i \frac{\partial^2 U_0}{\partial x \partial z} + 2p_i k_i \frac{\partial^2 U_0}{\partial z \partial y} \right) + \dots, \quad (6)$$

where $x_0, (x_0, y_0)$ or (x_0, y_0, z_0) are the coordinates of the central node, $x_i, (x_i, y_i)$ or (x_i, y_i, z_i) are the coordinates of the i th node in the star, and $h_i = x_i - x_0, k_i = y_i - y_0, p_i = z_i - z_0$.

If in Eqs. (4), (5) or (6) the terms over the second order are ignored, an approximation of second order for the U_i function is obtained. This is indicated as u_i . It is then possible to define the functions $B_2(u)$ in 1D, $B_5(u)$ in 2D or $B_9(u)$ in 3D as in [1,10]

$$B_2(u) = \sum_{i=1}^N \left[\left(u_0 - u_i + h_i \frac{\partial u_0}{\partial x} + \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} \right) w(h_i) \right]^2, \quad (7)$$

$$B_5(u) = \sum_{i=1}^N \left[\left(u_0 - u_i + h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{k_i^2}{2} \frac{\partial^2 u_0}{\partial y^2} + h_i k_i \frac{\partial^2 u_0}{\partial x \partial y} \right) w(h_i, k_i) \right]^2, \quad (8)$$

$$B_9(u) = \sum_{i=1}^N \left[\left(u_0 - u_i + h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} + \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{k_i^2}{2} \frac{\partial^2 u_0}{\partial y^2} + \frac{p_i^2}{2} \frac{\partial^2 u_0}{\partial z^2} + h_i k_i \frac{\partial^2 u_0}{\partial x \partial y} + h_i p_i \frac{\partial^2 u_0}{\partial x \partial z} + p_i k_i \frac{\partial^2 u_0}{\partial z \partial y} \right) w(h_i, k_i, p_i) \right]^2, \quad (9)$$

where $w(h_i)$, $w(h_i, k_i)$ and $w(h_i, k_i, p_i)$ are the denominated weight functions in 1D, 2D, or 3D, respectively.

If the norms (7), (8) or (9) are minimized with respect to the partial derivatives, the following linear equation systems are obtained:

$$\mathbf{A}_2 \mathbf{D}_{u2} = \mathbf{b}_2, \quad (10)$$

$$\mathbf{A}_5 \mathbf{D}_{u5} = \mathbf{b}_5, \quad (11)$$

$$\mathbf{A}_9 \mathbf{D}_{u9} = \mathbf{b}_9, \quad (12)$$

The matrices \mathbf{A}_2 , \mathbf{A}_5 and \mathbf{A}_9 are of 2×2 , 5×5 and 9×9 , respectively, and the vectors \mathbf{D}_{u2} , \mathbf{D}_{u5} and \mathbf{D}_{u9} are given, respectively, by

$$\mathbf{D}_{u2} = \left\{ \frac{\partial u_0}{\partial x}, \frac{\partial^2 u_0}{\partial x^2} \right\}^T, \quad (13)$$

$$\mathbf{D}_{u5} = \left\{ \frac{\partial u_0}{\partial x}, \frac{\partial u_0}{\partial y}, \frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_0}{\partial y^2}, \frac{\partial^2 u_0}{\partial x \partial y} \right\}^T, \quad (14)$$

$$\mathbf{D}_{u9} = \left\{ \frac{\partial u_0}{\partial x}, \frac{\partial u_0}{\partial y}, \frac{\partial u_0}{\partial z}, \frac{\partial^2 u_0}{\partial x^2}, \frac{\partial^2 u_0}{\partial y^2}, \frac{\partial^2 u_0}{\partial z^2}, \frac{\partial^2 u_0}{\partial x \partial y}, \frac{\partial^2 u_0}{\partial x \partial z}, \frac{\partial^2 u_0}{\partial z \partial y} \right\}^T. \quad (15)$$

From the previously obtained matrix equations (10)–(12) and by the fact that the matrices of coefficients \mathbf{A}_2 , \mathbf{A}_5 and \mathbf{A}_9 are symmetrical, it is possible to use the Cholesky method to solve the systems. The aim is to obtain the decomposition in upper and lower triangular matrices:

$$\mathbf{A}_2 = \mathbf{L}_2 \mathbf{L}_2^T, \quad (16)$$

$$\mathbf{A}_5 = \mathbf{L}_5 \mathbf{L}_5^T, \quad (17)$$

$$\mathbf{A}_9 = \mathbf{L}_9 \mathbf{L}_9^T. \quad (18)$$

The coefficients of the matrices \mathbf{L}_2 , \mathbf{L}_5 and \mathbf{L}_9 , are denoted by $l(i, j)$ with $i, j = 1, \dots, P$ and $P = 2, 5, 9$.

On solving systems (10)–(12), the following explicit difference formulae are obtained, with $P = 2$ for 1D case, $P = 5$ for 2D case and $P = 9$ for 3D case, [1]

$$\mathbf{D}_u(k) = \frac{1}{l(k, k)} \left(Y(k) - \sum_{i=1}^{P-k} l(k+i, k) D_u(k+i) \right) \quad (k = 1, \dots, P), \quad (19)$$

$$Y(k) = \left(-u_0 \sum_{i=1}^P M(k, i) c_i + \sum_{j=1}^N u_j \left(\sum_{i=1}^P M(k, i) d_{ji} \right) \right) \quad (k = 1, \dots, P),$$

$$M(i, j) = (-1)^{i+j} \frac{1}{l(i, i)} \sum_{k=j}^{i-1} l(i, k) M(k, j) \quad \text{with } j < i \quad (i, j = 1, \dots, P),$$

$$M(i, j) = \frac{1}{l(i, i)} \quad \text{with } j = i \quad (i, j = 1, \dots, P),$$

$$M(i, j) = 0 \quad \text{with } j > i \quad (i, j = 1, \dots, P),$$

$$c_i = \sum_{j=1}^N d_{ji}, \quad d_{j1} = h_j W^2, \quad d_{j2} = k_j W^2, \quad d_{j3} = p_j W^2, \quad d_{j4} = \frac{h_j^2}{2} W^2,$$

$$d_{j5} = \frac{k_j^2}{2} W^2, \quad d_{j6} = \frac{p_j^2}{2} W^2, \quad d_{j7} = h_j k_j W^2, \quad d_{j8} = h_j p_j W^2, \quad d_{j9} = p_j k_j W^2,$$

where

$$W^2 = (w(h_i))^2 \quad \text{or} \quad W^2 = (w(h_i, k_i))^2 \quad \text{or} \quad W^2 = (w(h_i, k_i, p_i))^2. \quad (20)$$

On including the explicit expressions for the values of the partial derivatives (19) in Eq. (3a), the star equation is obtained

$$\mathfrak{I}[u] = -m_0 u_0 + \sum_{i=1}^N m_i u_i = 0 \quad (21)$$

then

$$u_0 = \frac{1}{m_0} \sum_{i=1}^N m_i u_i \quad \text{with} \quad \sum_{i=1}^N m_i = m_0. \quad (22)$$

In order to clarify the above formulae (16), (21), (22), the development corresponding to the 1D case is included in Appendix A, and an example is given in Appendix B.

The application of the above procedure to each one of the nodes of the mesh, gives us a system of linear equations. On solving this system of equations we are provided with approximated values of the function u in the nodes of the domain. This approximated value has been obtained by the GFDM.

2.2. Explicit method

Secondly, we shall use an explicit formula for the part of Eqs. (1a) and (2a) that depends on time. This explicit formula can be used to solve the Cauchy initial value problem. This method involves only one grid point at the advanced time level.

The first derivative of u with respect to time is approached using the explicit method by the forward difference formula:

$$\frac{\partial u}{\partial t} = \frac{u_0^{n+1} - u_0^n}{\Delta t} \quad (23)$$

and the second derivative with respect to time by

$$\frac{\partial^2 u}{\partial t^2} = \frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{(\Delta t)^2}. \quad (24)$$

In the following paragraph the complete approximations of Eqs. (1a) and (2a) taking into account (21), (23), (24) are given.

3. Explicit differences schemes

3.1. Parabolic equations

If the values (21) and the forward difference formula for the derivative of u with respect to time (23) are then substituted in Eq. (1a) the following expression is obtained:

$$u_0^{n+1} = u_0^n(1 - m_0\Delta t) + \Delta t \left(\sum_{i=1}^N m_i u_i^n \right). \quad (25)$$

The expression (25) relates the value of the function at the central node of the star, in time $n + 1$, with the values of the functions in the nodes of the star, for a time n , multiplied by specific coefficients. This then indicates that the value of the function in a time $n + 1$ is a weighting sum of the values of the function in the star for the time n .

3.1.1. Hyperbolic equations

If the expressions (21) and (24) are substituted in Eq. (2a) the following recursive relationship is obtained:

$$u_0^{n+1} = (\Delta t)^2 c^2 \left(\sum_{i=1}^N m_i u_i^n \right) + u_0^n(2 - m_0 c^2 (\Delta t)^2) - u_0^{n-1}. \quad (26)$$

The first derivative with respect to the time is approached by the central difference formula

$$\frac{\partial u}{\partial t} = \frac{u_0^1 - u_0^{-1}}{2\Delta t} = h(\{x_0\}). \quad (27)$$

Then

$$u_0^{-1} = u_0^1 - 2\Delta t h(\{x_0\}). \quad (28)$$

If (28) is substituted in Eq. (26), the following expression is obtained:

$$u_0^1 = \frac{(\Delta t)^2 c^2 \left[\sum_{i=1}^N m_i u_i^0 \right] + [2 - m_0 c^2 (\Delta t)^2] u_0^0}{2} + h(\{x_0\}) \Delta t. \quad (29)$$

The expression (29) relates the value of the function at the central node of the star, at time $n = 1$, with the values of the functions in the nodes of the star for a time $n = 0$, and the initial condition $h(\{x_0\})$.

4. Convergence

According to the Lax's equivalence theorem, if the consistency condition is satisfied, stability is the necessary and sufficient condition for convergence. In this section we study firstly the truncation error of parabolic (1) and hyperbolic (2) equations, and secondly consistency and the stability.

4.1. Truncation error

We can split the total truncation error of parabolic and hyperbolic equations previously defined in (1) and (2) in two parts, the first one corresponding to time derivatives, and the second one corresponding to the space derivatives contained in the Ψ operator of (1) and (2).

As it is well known, the truncation errors for first and second order time derivatives (TE_t) are given as follows:

$$\frac{\partial u(\{x\}, t)}{\partial t} = \frac{u(\{x_0\}, t + \Delta t) - u(\{x_0\}, t)}{\Delta t} - \frac{\Delta t}{2} \frac{\partial^2 u(\{x_0\}, t_1)}{\partial t^2} + O((\Delta t)^2), \quad t < t_1 < t + \Delta t, \quad (30)$$

$$(TE_t)_{\text{first derivative}} = -\frac{\Delta t}{2} \frac{\partial^2 u(\{x_0\}, t_1)}{\partial t^2} + O((\Delta t)^2), \quad t < t_1 < t + \Delta t, \quad (31)$$

$$\begin{aligned} \frac{\partial^2 u(\{x\}, t)}{\partial t^2} &= \frac{u(\{x_0\}, t + \Delta t) - 2u(\{x_0\}, t) + u(\{x_0\}, t - \Delta t)}{(\Delta t)^2} \\ &\quad - \frac{(\Delta t)^2}{12} \frac{\partial^4 u(\{x_0\}, t_1)}{\partial t^4} + O((\Delta t)^4), \quad t < t_1 < t + \Delta t, \end{aligned} \quad (32)$$

$$(TE_t)_{\text{second derivative}} = -\frac{(\Delta t)^2}{12} \frac{\partial^4 u(\{x_0\}, t_1)}{\partial t^4} + O((\Delta t)^4), \quad t < t_1 < t + \Delta t. \quad (33)$$

In order to obtain the truncation error for space derivatives, Taylor's series expansion including higher order derivatives is used and then higher order functions $B_p^*[u]$, $p = 2, 5, 9$ are obtained. The expressions of $B_p^*[u]$, $p = 2, 5, 9$ are similar to the ones given in (7)–(9), but incorporating now higher order derivatives. If the new norms $B_p^*[u]$, $p = 2, 5, 9$ are minimized with respect to the partial derivatives down to the second order, the following linear equation systems are obtained:

$$A_p D_{u_p} = b_p^*, \quad (34)$$

where A_p and D_{u_p} ($p = 2, 5, 9$) are as calculated previously in (10)–(12), and b_p^* ($p = 2, 5, 9$) can be split in two parts as follows:

$$b_p^* = b_p + b_p^{**}, \quad (35)$$

where b_p ($p = 2, 5, 9$) are as previously calculated in (10)–(12) and the new terms b_p^{**} correspond to the new higher order derivatives incorporated in the Taylor's series expansion to extend the functions from $B_p[u]$, $p = 2, 5, 9$ to $B_p^*[u]$, $p = 2, 5, 9$.

Then a better approximation of the partial derivatives can be obtained using the inverse matrix A_p^{-1}

$$D_{u_p} = A_p^{-1} b_p + A_p^{-1} b_p^{**}, \quad (36)$$

where D_{u_p} includes the same derivatives as given previously in (13)–(15); $A_p^{-1} b_p$ is the approximation used in the GFD method and then the truncation errors for derivatives are given by $A_p^{-1} b_p^*$.

Let us consider now, that the operator $\Psi[u]$ of (1a) or (2a) is defined by

$$\Psi[u] = (\gamma_1, \gamma_2, \dots, \gamma_p) D_{u_p}, \quad (37)$$

where $\gamma_1, \dots, \gamma_p$ are the constant coefficients of the space partial derivatives, then the truncation error of the spatial derivatives (TE_p) is given by

$$\begin{aligned} \Psi[u] &= (\gamma_1, \gamma_2, \dots, \gamma_p) D_{u_p} = (\gamma_1, \gamma_2, \dots, \gamma_p) A_p^{-1} b_p + (\gamma_1, \gamma_2, \dots, \gamma_p) A_p^{-1} b_p^* \\ \Rightarrow \Psi[u] - (\gamma_1, \gamma_2, \dots, \gamma_p) A_p^{-1} b_p &= (\gamma_1, \gamma_2, \dots, \gamma_p) A_p^{-1} b_p^* = TE_p \end{aligned} \quad (38)$$

and taking into account (21)

$$(\gamma_1, \gamma_2, \dots, \gamma_p) A_p^{-1} b_p = \mathfrak{I}[u] = -m_0 u_0 + \sum_{i=1}^N m_i u_i. \quad (39)$$

Then we can establish the truncation error for spatial derivatives. We develop only the truncation error corresponding to 1D case ($p = 2$). The other truncation errors for 2D ($p = 5$) and 3D ($p = 9$) are developed in Appendix C.

For ($p = 2$) the extended function $B_2^*[u]$ is given by

$$B_2^*[u] = \sum_{i=1}^N \left[\left(u_0 - u_i + h_i \frac{\partial u_0}{\partial x} + \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{h_i^3}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \dots \right) w(h_i) \right]^2 \quad (40)$$

and the system of equations (34)

$$\begin{pmatrix} \sum_{i=1}^N h_i^2 w^2(h_i) & \sum_{i=1}^N \frac{h_i^3}{2} w^2(h_i) \\ \sum_{i=1}^N \frac{h_i^3}{2} w^2(h_i) & \sum_{i=1}^N \frac{h_i^4}{4} w^2(h_i) \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 u_0}{\partial x^2} \end{pmatrix} \\ = \begin{pmatrix} \sum_{i=1}^N (-u_0 + u_i) h_i w^2(h_i) - \frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \frac{h_i^4}{6} w^2(h_i) - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \frac{h_i^5}{24} w^2(h_i) - \dots \\ \sum_{i=1}^N (-u_0 + u_i) \frac{h_i^2}{2} w^2(h_i) - \frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \frac{h_i^5}{12} w^2(h_i) - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \frac{h_i^6}{48} w^2(h_i) - \dots \end{pmatrix} \quad (41)$$

where $N \geq 2$, and then

$$\text{TE}_{p=2} = (\gamma_1, \gamma_2) A_2^{-1} b_2^* = (\gamma_1, \gamma_2) A_2^{-1} \begin{pmatrix} -\frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \frac{h_i^4}{6} w^2 - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \frac{h_i^5}{24} w^2 - \dots \\ -\frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \frac{h_i^5}{12} w^2 - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \frac{h_i^6}{48} w^2 - \dots \end{pmatrix} \quad (42)$$

and operating

$$\text{TE}_{p=2} = -\frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N m_i \frac{h_i^3}{6} - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N m_i \frac{h_i^4}{24} - \dots \quad (43)$$

From (100) of Appendix A, m_i coefficients can be expressed by

$$m_i = \frac{\mu_i}{h_i^2}, \quad \mu_i \in \mathbb{R} \quad (44)$$

and by substituting (44) in (43)

$$\text{TE}_{p=2} = -\frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \mu_i \frac{h_i}{6} - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \mu_i \frac{h_i^2}{24} + O(h_i^3) \quad (45)$$

which is the truncation error for spatial derivative in 1D case.

Taking into account that the total truncation errors (TTE) for parabolic or hyperbolic equations are given by ($\text{TTE} = \text{TE}_t + \text{TE}_p$), the following total truncation errors are obtained:

(a) Parabolic equation:

$$\text{TTE}(\text{parabolic}) = -\frac{\Delta t}{2} \frac{\partial^2 u(\{x_0\}, t_1)}{\partial t^2} - \frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \mu_i \frac{h_i}{6} - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \mu_i \frac{h_i^2}{24} + O(h_i^3) + O((\Delta t)^2). \quad (46)$$

(b) Hyperbolic equation:

$$\text{TTE}(\text{hyperb}) = -\frac{(\Delta t)^2}{12} \frac{\partial^4 u(\{x_0\}, t_1)}{\partial t^4} - \frac{\partial^3 u_0}{\partial x^3} \sum_{i=1}^N \mu_i \frac{h_i}{6} - \frac{\partial^4 u_0}{\partial x^4} \sum_{i=1}^N \mu_i \frac{h_i^2}{24} + O(h_i^3) + O((\Delta t)^4). \quad (47)$$

4.2. Consistency

By considering bounded derivatives in (46) and (47)

$$\lim_{(\Delta t, h_i) \rightarrow 0} \text{TTE}(\text{parabolic}) \rightarrow 0, \quad (48)$$

$$\lim_{(\Delta t, h_i) \rightarrow 0} \text{TTE}(\text{hyperb}) \rightarrow 0. \quad (49)$$

Then, the truncation errors given in (46) and (47) show the consistency of the approximation for parabolic and hyperbolic equations with constant coefficients.

In the following paragraph the stability is studied using the von Neumann criterium.

4.3. Stability criteria

For the difference schemes, the von Neumann condition is sufficient as well as necessary for stability [7]. Boundary conditions are neglected by the von Neumann method which applies in theory only to pure initial value problems with periodic initial data. It does however provide necessary conditions for stability of constant coefficient problems regardless of the type of boundary condition [7].

For the stability analysis a harmonic decomposition is made of the approximate solution at grid points at a given time level. Then, by following the von Neumann idea for stability analysis, we can write that the finite difference approximation in the central node at time n , may be expressed as

$$u_0^n = \xi^n e^{i\{\alpha_k\}^T \{x_0\}} \quad (50)$$

and the finite difference approximation in the other nodes of the star

$$u_j^n = \xi^n e^{i\{\alpha_k\}^T \{x_j\}}, \quad (51)$$

where $\{\alpha_k\}$, $k = 1, 2, 3$ is the column vector of the wave numbers, $\{x_0\}$ is the vector of coordinates of central node of star and $\{x_j\}$ is the vector of coordinates of the other nodes of star, being

$$\{x_j\} = \{x_0\} + \{h_j\} \quad (52)$$

then $\{h_j\}$ are the relative coordinates between the nodes of star and the central node.

On the other hand, ξ is called the amplification factor and it is in general a complex constant. If this amplification factor has a modulus greater than unity ($|\xi| > 1$) the method is unstable.

4.3.1. Parabolic equations

Substituting (50) and (51) into (25), we obtain

$$\xi^{n+1} e^{i\{\alpha_k\}^T \{x_0\}} = \xi^n e^{i\{\alpha_k\}^T \{x_0\}} (1 - \Delta t m_0) + \Delta t \sum_{j=1}^N m_j \xi^n e^{i\{\alpha_k\}^T \{x_j\}}. \quad (53)$$

Using (52), cancellation of $\xi^n e^{i\{\alpha_k\}^T \{x_0\}}$, leads to

$$\xi = (1 - \Delta t m_0) + \Delta t \sum_{j=1}^N m_j e^{i\{\alpha_k\}^T \{h_j\}} \quad (54)$$

and as we know by (22) that

$$m_0 = \sum_{j=1}^N m_j. \quad (55)$$

Substituting (55) into (54) we obtain

$$\xi = 1 - \Delta t \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right). \quad (56)$$

The modulus of the amplification factor is

$$|\xi| = \left| 1 - \Delta t \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \right|. \quad (57)$$

If we consider now the condition for stability

$$|\xi| \leq 1 \quad (58)$$

it is possible to write

$$\begin{aligned} -1 &\leq 1 - \Delta t \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \leq 1 \Rightarrow -2 \leq -\Delta t \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \leq 0 \\ &\Rightarrow 0 \leq \Delta t \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \leq 2. \end{aligned} \quad (59)$$

As we know

$$\begin{aligned} |1 - e^{i\{\alpha_k\}^T \{h_j\}}| &= \sqrt{(1 - \cos\{\alpha_k\}^T \{h_j\})^2 + (-\sin\{\alpha_k\}^T \{h_j\})^2} \\ &= \sqrt{2 - 2\cos\{\alpha_k\}^T \{h_j\}} = |2\sin(2\{\alpha_k\}^T \{h_j\})| \leq 2. \end{aligned} \quad (60)$$

Then we can write the condition (59) as

$$0 \leq 2\Delta t \sum_{j=1}^N m_j \leq 2 \Rightarrow 0 \leq \Delta t \sum_{j=1}^N m_j \leq 1 \Rightarrow 0 \leq \Delta t \leq \frac{1}{\sum_{j=1}^N m_j}. \quad (61)$$

And the condition for stability of parabolic equations is

$$0 \leq \Delta t \leq \frac{1}{|m_0|_{\max}} \quad (62)$$

being possible to define Δt . Note that $|m_0|_{\max}$ increases with the dimension of the space considered.

Thus, convergence is assured for parabolic partial differential equation with constant coefficients provided that carry out condition (62). The convergence is of first order in time according to (46).

4.3.2. Hyperbolic equations

Substituting (50) and (51) into (26), the following expression is obtained

$$\xi^{n+1} e^{i\{\alpha_k\}^T \{x_0\}} = \xi^n e^{i\{\alpha_k\}^T \{x_0\}} (2 - (\Delta t)^2 c^2 m_0) + (\Delta t)^2 c^2 \left(\sum_{j=1}^N m_j \xi^n e^{i\{\alpha_k\}^T \{x_j\}} \right) - \xi^{n-1} e^{i\{\alpha_k\}^T \{x_0\}}. \quad (63)$$

Using (52), cancellation of $\xi^n e^{i\{\alpha_k\}^T \{x_0\}}$ leads to

$$\xi = (2 - (\Delta t)^2 c^2 m_0) + (\Delta t)^2 c^2 \left(\sum_{j=1}^N m_j e^{i\{\alpha_k\}^T \{h_j\}} \right) - \xi^{-1}. \quad (64)$$

Using (58) and after some calculus we obtain the quadratic equation

$$\xi^2 - \left(2 - (\Delta t)^2 c^2 \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \right) \xi + 1 = 0. \quad (65)$$

Hence the values of ξ are

$$\xi = b \pm \sqrt{b^2 - 1}, \quad (66)$$

where

$$b = \frac{1}{2} \left(2 - (\Delta t)^2 c^2 \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \right)$$

If we consider now the condition for stability (58), we obtain

$$|b \pm \sqrt{b^2 - 1}| \leq 1. \quad (67)$$

If

$$b > 1 \Rightarrow |\xi_1| > 1 \Rightarrow \text{giving instability},$$

$$b < -1 \Rightarrow |\xi_2| > 1 \Rightarrow \text{giving instability}$$

and if

$$-1 \leq b \leq 1 \Rightarrow |\xi_1| = |\xi_2| = |b \pm i\sqrt{1 - b^2}| = 1.$$

Then the condition is

$$-1 \leq b \leq 1 \Rightarrow -1 \leq \left(1 - \frac{1}{2} (\Delta t)^2 c^2 \sum_{j=1}^N m_j \left(1 - e^{i\{\alpha_k\}^T \{h_j\}} \right) \right) \leq 1 \quad (68)$$

and if we consider (55) and the condition (60) is possible to write

$$-1 \leq 1 - (\Delta t)^2 c^2 m_0 \leq 1 \Rightarrow -2 \leq -(\Delta t)^2 c^2 m_0 \leq 0 \Rightarrow 0 \leq (\Delta t)^2 c^2 m_0 \leq 2. \quad (69)$$

And the condition for stability of parabolic equations is

$$0 \leq \Delta t \leq \sqrt{\frac{2}{c^2 |m_0|_{\max}}}. \quad (70)$$

Note that $|m_0|_{\max}$ increases with the dimension of the space considered.

Thus, convergence is assured for hyperbolic partial differential equation with constant coefficients provided that carry out condition (70). The convergence is of second order in time according to (47).

5. Numerical results

This section provides the numerical results obtained when solving partial derivative equations in time dependent (parabolic and hyperbolic), using the GFDM and the classical explicit method as proposed in this paper. Two sets of examples (the first one corresponding to parabolic equations and the second one corresponding to hyperbolic equations) are given. In each set, three different cases corresponding to 1D, 2D and 3D are studied. In all the examples considered in this paper, the weighting function used has been

$$w = \frac{1}{(\text{dist})^3}, \quad (71)$$

where “(dist)” is the Euclidean distance from each node of the star to the central node.

By knowing the analytical solution, two different errors have been calculated—the global error and the maximum local error. The global error is evaluated for each time increment, in the last time step considered, using the following formula:

$$\text{Global error} = \frac{\sqrt{\frac{\sum_{i=1}^{NT} (\text{sol}(i) - \text{exac}(i))^2}{NT}}}{|\text{exac}_{\max}|} \times 100. \quad (72)$$

The maximum local error is evaluated for each time step using the following formula:

$$\text{Maximum local error} = \max_i |\text{sol}(i) - \text{exac}(i)|, \quad (73)$$

where $\text{sol}(i)$ is the GFDM solution at the node “ i ”. $\text{Exac}(i)$ is the exact value of the solution at the node “ i ”. Exac_{\max} is the maximum value of the exact values in the cloud of nodes considered and NT is the total number of nodes of the domain considered.

5.1. Parabolic equations

We consider three examples corresponding to 1D, 2D and 3D cases, respectively.

5.1.1. 1D case

Let us consider equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2 - 6x, \quad t > 0, \quad 0 < x < 1 \quad (74)$$

with the initial condition

$$u(x, 0) = \sin \pi x + x(1 - x)^2 \quad \text{in } 0 \leq x \leq 1 \quad (75)$$

and boundary conditions

$$u(0, t) = u(1, t) = 0. \quad (76)$$

The exact solution is

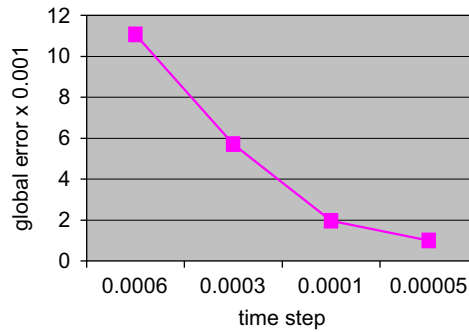
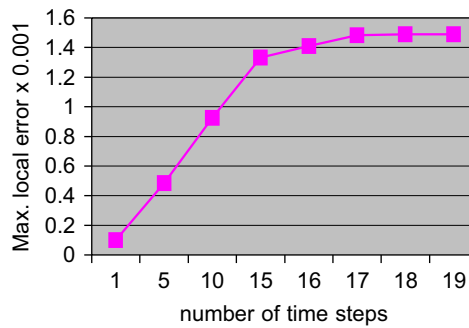
$$u(x, t) = e^{-\pi^2 t} \sin(\pi x) + x(1 - x)^2 \quad (77)$$

and the following grid or cloud of nodes given in Fig. 2, has been used.

The influence over the global error of using different values of the time increment is given in Fig. 3 considering $\Delta t = 0.00005$. In Fig. 4, the variation of the maximum local error of the points of the domain after a number of time steps is shown.



Fig. 2. Irregular grid of nodes (21 nodes).

Fig. 3. Variation of the global error in the last time step calculated versus the time step size (Δt).Fig. 4. Maximum local error versus the number of time steps (n), for $\Delta t = 0.00005$.

5.1.2. 2D case

Let us consider equation

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad t > 0, \quad 0 < x < 1; \quad 0 < y < 1 \quad (78)$$

with the initial condition

$$u(x, y, 0) = \sin \pi \left(\frac{x+y}{2} \right) \quad 0 \leq x \leq 1; \quad 0 \leq y \leq 1 \quad (79)$$

and boundary conditions

$$\left. \begin{aligned} \frac{\partial u(0, y, t)}{\partial x} &= \frac{\pi}{2} e^{-\pi^2 t/2} \cos \pi \left(\frac{y}{2} \right) \\ \frac{\partial u(1, y, t)}{\partial x} &= \frac{\pi}{2} e^{-\pi^2 t/2} \cos \pi \left(\frac{1+y}{2} \right) \\ u(x, 0, t) &= e^{-\pi^2 t/2} \sin \pi \left(\frac{x}{2} \right) \\ u(x, 1, t) &= e^{-\pi^2 t/2} \sin \pi \left(\frac{x+1}{2} \right) \end{aligned} \right\} \begin{aligned} &0 < y < 1, \quad t > 0 \\ &0 \leq x \leq 1, \quad t > 0 \end{aligned} \quad (80)$$

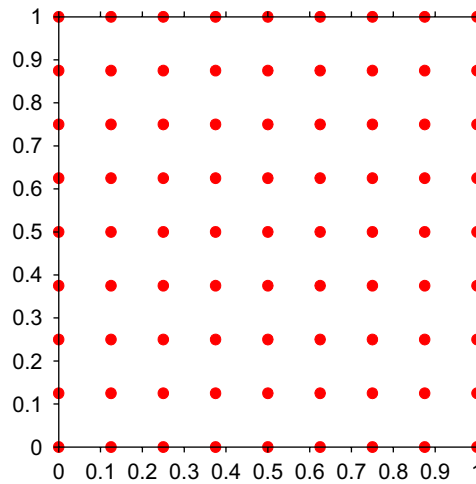


Fig. 5. Grid of nodes (regular).

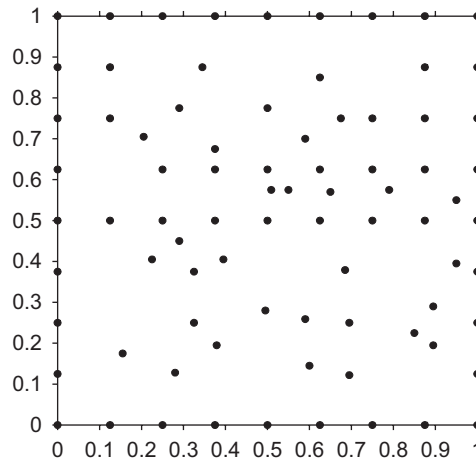


Fig. 6. Grid of nodes (irregular).

the exact solution is

$$u(x, y, t) = e^{-\pi^2 t/2} \sin \pi \left(\frac{x+y}{2} \right) \quad (81)$$

In this example, we consider the use of regular or irregular grids. The influence on global error of using different values of time increment and different grids of nodes, Figs. 5 (regular), 6 (irregular) and 7 (irregular) is given in Figs. 8–10, respectively.

The results obtained with the regular grid of nodes (Fig. 5) are shown in Fig. 8. Two different irregular grids, both with the same number of nodes are shown in Figs. 6 and 7. The results obtained corresponding to the irregular grids of Figs. 6 and 7 are shown in Figs. 9 and 10, respectively.

By comparing the results obtained with a totally random irregular grid (Fig. 6) and the regular cloud of Fig. 5, we can see that the regular grid is more accurate, see Figs. 8 and 9. The irregularity of clouds of nodes can be taken as totally at random (Fig. 6). But it can also be established according to the analytical solution (81) as it has been made to generate the grid of Fig. 7. Better accuracy can be obtained with the irregular grid as compared with the one obtained with the regular grid, see the errors obtained in Figs. 8 and 10.

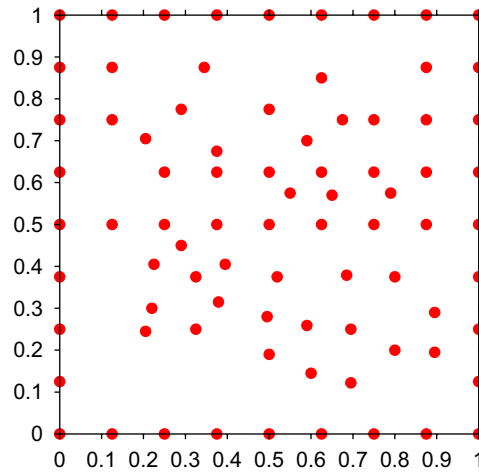
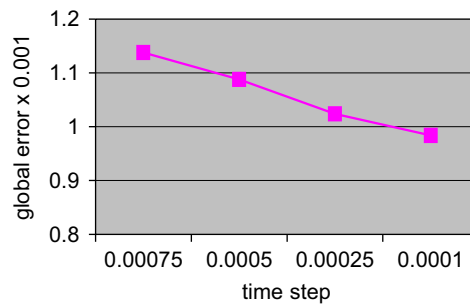
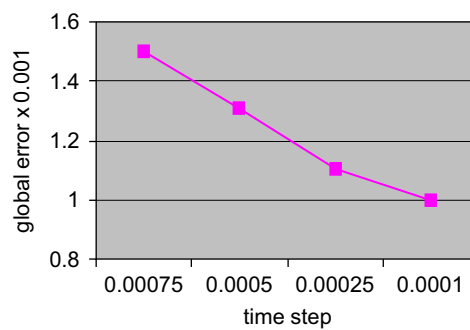


Fig. 7. Grid of nodes (irregular).

Fig. 8. Variation of the global error in the last time step calculated versus the time step size (Δt) (regular grid (Fig. 5)).Fig. 9. Variation of the global error in the last time step calculated versus the time step size (Δt) (irregular grid (Fig. 6)).

5.1.3. 3D case

Let us consider equation

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad t > 0, \quad 0 < x < 1; \quad 0 < y < 1; \quad 0 < z < 1 \quad (82)$$

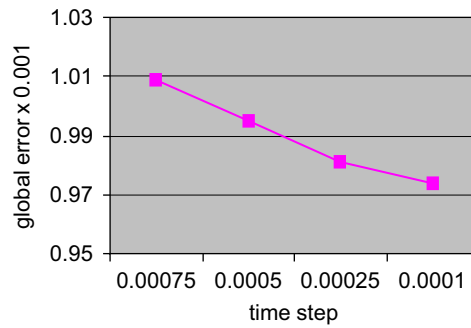


Fig. 10. Variation of the global error in last time step calculated versus the time step (Δt) (irregular grid (Fig. 7)).

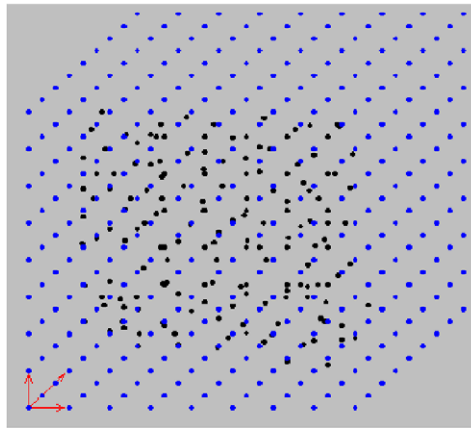


Fig. 11. Grid of nodes.

with Dirichlet boundary conditions and the initial condition

$$u(x, y, z, 0) = \sin \pi \frac{x + y + z}{3} \quad \text{in } 0 \leq x, y, z \leq 1 \quad (83)$$

the exact solution is

$$u(x, y, z, t) = e^{-(\pi^2/3)t} \sin \pi \frac{x + y + z}{3} \quad (84)$$

in the grid or cloud of nodes of Fig. 11.

The influence on the global error by using different values of the time increment is given in Fig. 12. In Fig. 13, the variation of the maximum local error of the points of the domain after a number of time steps is shown.

5.2. Hyperbolic equations

We consider three examples corresponding to 1D, 2D and 3D cases, respectively.

5.2.1. 1D case

Let us consider equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < 1 \quad (85)$$

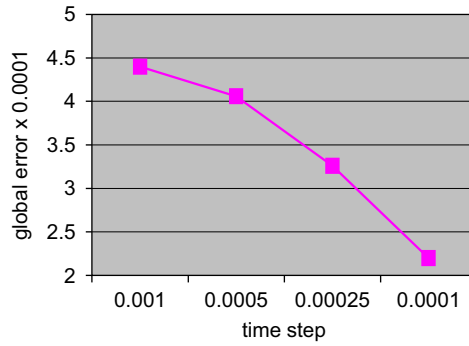


Fig. 12. Variation of the global error in the last time step calculated versus the time step size (Δt).

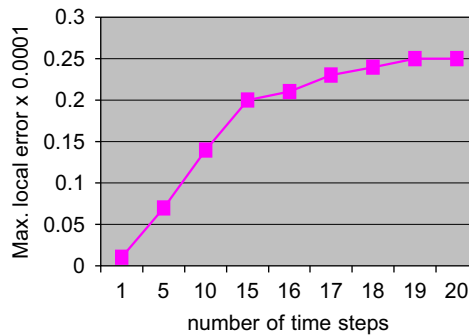


Fig. 13. Maximum local error versus the number of time step (n), for $\Delta t = 0.0001$.



Fig. 14. Grid of nodes.

with the initial conditions

$$\begin{cases} u(x, 0) = \sin 2\pi x \\ \frac{\partial u(x, 0)}{\partial t} = 2\pi \sin 2\pi x \end{cases} \quad \text{in } 0 \leq x \leq 1 \quad (86)$$

and boundary conditions

$$u(0, t) = u(1, t) = 0 \quad (87)$$

the exact solution is

$$u(x, t) = \sin 2\pi x (\cos 2\pi t + \sin 2\pi t) \quad (88)$$

in the grid or cloud of nodes of Fig. 14.

The results obtained are given in Figs. 15 and 16.

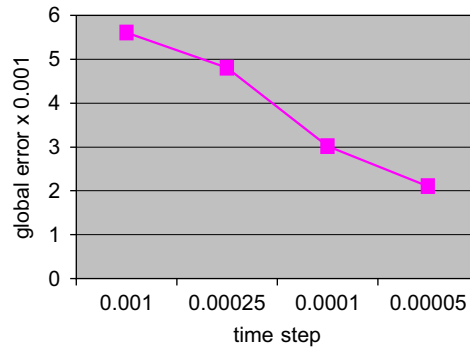


Fig. 15. Variation of the global error in the last time step calculated versus the time step size (Δt).

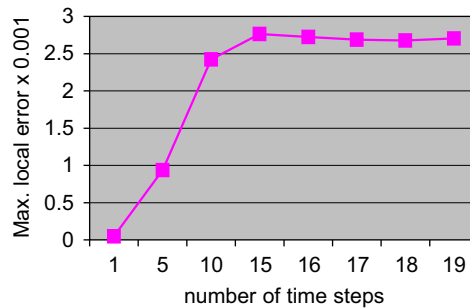


Fig. 16. Maximum local error versus the number of time step (n), for $\Delta t = 0.00005$.

5.2.2. 2D case

Let us consider equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad t > 0, \quad 0 < x, y < 1 \quad (89)$$

with the initial conditions

$$\begin{cases} u(x, y, 0) = \sin \pi x \sin \pi y + x^2 - y^2 \\ \frac{\partial u(x, y, 0)}{\partial t} = 0 \end{cases} \quad \text{in } 0 \leq x, y \leq 1 \quad (90)$$

and Dirichlet boundary conditions, the exact solution is

$$u(x, y, t) = \cos(\sqrt{2}\pi t) \sin \pi x \sin \pi y + x^2 - y^2. \quad (91)$$

The grids used are the same grids used before for the parabolic 2D Eq. (78) given in Figs. 5 and 6. The influence over global error by using different values of time increment and different clouds of nodes, Figs. 6 (irregular) and 5 (regular) are shown in Figs. 17. and 18

The irregularity of the cloud of nodes in Fig. 6 applied to the hyperbolic equation (89) has been considered as totally at random then, in this case better accuracy has been obtained with the regular grid.

5.2.3. 3D case

Let us consider equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad t > 0, \quad 0 < x, y, z < 1 \quad (92)$$

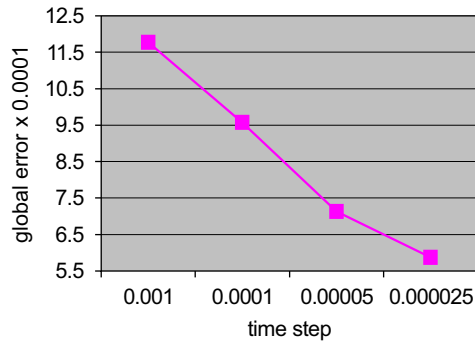


Fig. 17. Variation of the global error in the last time step calculated versus the time step size (Δt) (irregular grid of nodes).

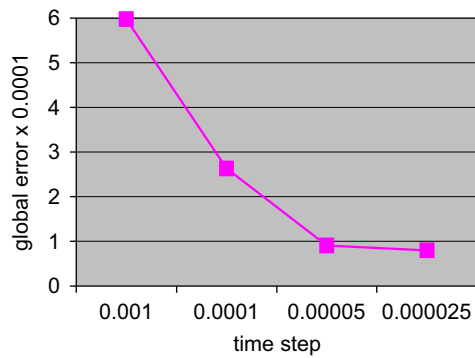


Fig. 18. Variation of the global error in the last time step calculated versus the time step (Δt) (regular grid of nodes).

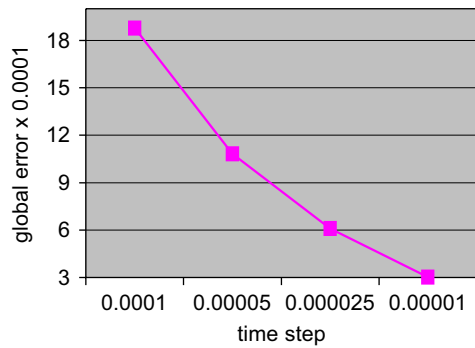


Fig. 19. Variation of the global error in the last time step calculated versus the time step size (Δt).

with Dirichlet boundary conditions and the initial conditions

$$\begin{cases} u(x, y, z, 0) = \sin \pi x \sin \pi y \sin \pi z \\ \frac{\partial u(x, y, z, 0)}{\partial t} = 0 \end{cases} \quad \text{in } 0 \leq x, y, z \leq 1 \quad (93)$$

the exact solution is

$$u(x, y, t) = \cos \sqrt{3}\pi t \sin \pi x \sin \pi y \sin \pi z \quad (94)$$

in the same grid of nodes previously shown in Fig. 11. The results obtained are given in Figs. 19 and 20.

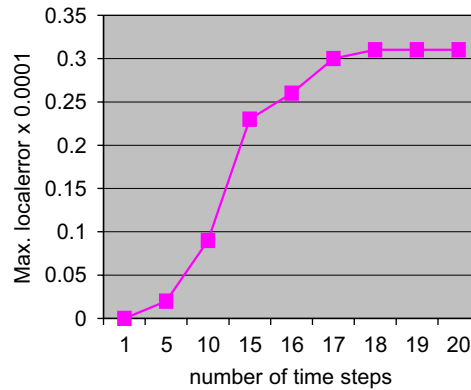


Fig. 20. Maximum local error versus number of time step (n), for $\Delta t = 0.00001$.

6. Conclusions

The use of the generalized finite difference (GFD) method using irregular clouds of points is important for solving partial differential equations. The extension of the GFD to the explicit solution of parabolic and hyperbolic equations has been developed.

The truncation errors of parabolic and hyperbolic equations in the case of irregular grids of points have been defined. The von Neumann stability criterion has been expressed in function of the coefficients of the star equation for irregular clouds of nodes. This all generalizes the existing results of the stability limit for the explicit method using regular grids to the more general case of using irregular grids.

Different examples have been solved using the explicit finite difference formulae and the criterion of stability, in one, two or three space dimensions. The numerical results obtained in 2D cases studied show the accuracy that can be obtained over irregular versus regular grids. As is shown in the numerical results, a decrease in the value of the time step, always below the stability limit (or critical time step), leads to a decrease of the global error. Also in the results obtained how the finite difference approximation values remain stable when the number of time steps is increased is shown.

Acknowledgement

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Appendix A

In the 1D case matrix \mathbf{A}_2 is given by

$$\mathbf{A}_2 = \begin{pmatrix} \sum_{i=1}^N h_i^2 w^2(h_i) & \sum_{i=1}^N \frac{h_i^3}{2} w^2(h_i) \\ \sum_{i=1}^N \frac{h_i^3}{2} w^2(h_i) & \sum_{i=1}^N \frac{h_i^4}{4} w^2(h_i) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (95)$$

then Eq. (10) is

$$\mathbf{A}_2 \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 u_0}{\partial x^2} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N (-u_0 + u_i) h_i w^2(h_i) \\ \sum_{i=1}^N (-u_0 + u_i) \frac{h_i^2}{2} w^2(h_i) \end{pmatrix}, \quad N \geq 2 \quad (96)$$

and Eq. (16) is

$$\mathbf{A}_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = L_2 L_2^T = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{pmatrix} \Rightarrow \begin{cases} l_{11} = \sqrt{a_{11}}, \\ l_{21} = \frac{a_{12}}{\sqrt{a_{11}}}, \\ l_{22} = \sqrt{a_{22} - l_{21}^2}, \end{cases} \quad (97)$$

$$\begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} \\ 0 & l_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 u_0}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^N (-u_0 + u_i) h_i w^2(h_i) \\ \sum_{i=1}^N (-u_0 + u_i) \frac{h_i^2}{2} w^2(h_i) \end{pmatrix}. \quad (98)$$

If for example we consider $\Psi[U] = C_1(\partial^2 U/\partial x^2) + C_2(\partial U/\partial x)$, with C_1, C_2 constants, by including the explicit expressions for the values of the partial derivatives, we have

$$\begin{aligned} \Psi[u] = & -u_0 \left(\frac{1}{l_{22}^2} \left[C_1 - C_2 \frac{l_{12}}{l_{11}} \right] \sum_{i=1}^N \frac{h_i^2}{2} w^2(h_i) + \left(-C_1 \frac{l_{12}}{l_{11} l_{22}^2} + C_2 \left(\frac{1}{l_{11}^2} + \frac{l_{12}^2}{l_{11} l_{22}^2} \right) \right) \sum_{i=1}^N h_i w^2(h_i) \right) \\ & + \sum_{i=1}^N \left(\frac{1}{l_{22}^2} \left[C_1 - C_2 \frac{l_{12}}{l_{11}} \right] \frac{h_i^2}{2} w^2(h_i) + \left(-C_1 \frac{l_{12}}{l_{11} l_{22}^2} + C_2 \left(\frac{1}{l_{11}^2} + \frac{l_{12}^2}{l_{11} l_{22}^2} \right) h_i w^2(h_i) \right) \right) u_i \end{aligned} \quad (99)$$

and taking into account (21), the coefficients $m_0, m_i (i = 1, \dots, N)$

$$\begin{cases} m_0 = \frac{1}{l_{22}^2} \left[C_1 - C_2 \frac{l_{12}}{l_{11}} \right] \sum_{i=1}^N \frac{h_i^2}{2} w^2(h_i) + \left(-C_1 \frac{l_{12}}{l_{11} l_{22}^2} + C_2 \left(\frac{1}{l_{11}^2} + \frac{l_{12}^2}{l_{11} l_{22}^2} \right) \right) \sum_{i=1}^N h_i w^2(h_i), \\ m_i = \frac{1}{l_{22}^2} \left[C_1 - C_2 \frac{l_{12}}{l_{11}} \right] \frac{h_i^2}{2} w^2(h_i) + \left(-C_1 \frac{l_{12}}{l_{11} l_{22}^2} + C_2 \left(\frac{1}{l_{11}^2} + \frac{l_{12}^2}{l_{11} l_{22}^2} \right) \right) h_i w^2(h_i) \end{cases} \quad (100)$$

that carries out condition $m_0 = \sum_{i=1}^N m_i$.

Appendix B

B.1. Example of 1D case

Let us consider the simple 1D case $\Psi[U] = \partial^2 U/\partial x^2$, in order to calculate the star equation for the nodal arrangement shown in Fig. B1, we consider two different cases both with the central node at 0 point and two nodes taken in the neighbourhood ($N = 2$):

(a) nodes of the star [1,2], (b) nodes of the star [2,3].

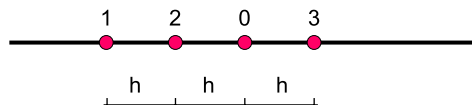


Fig. B1. Grid of nodes.

In both cases the weighting function used is

$$\frac{1}{(\text{dist})^3} = \frac{1}{|x_i - x_0|^3} \quad (101)$$

x_i, x_0 being the coordinates of the star nodes and the coordinate of the central node respectively.

Taking into account (95) and (96) we obtain:

Case (a):

$$\begin{pmatrix} \frac{17}{16h^4} & -\frac{9}{16h^3} \\ -\frac{9}{16h^3} & \frac{5}{16h^2} \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 u_0}{\partial x^2} \end{pmatrix} = \begin{pmatrix} \frac{33}{32h^5} u_0 - \frac{u_2}{h^5} - \frac{u_1}{32h^5} \\ -\frac{17}{32h^4} u_0 + \frac{u_2}{2h^4} + \frac{u_1}{32h^4} \end{pmatrix} \quad (102)$$

then the star equation (21) is

$$\Rightarrow \mathfrak{I}[u] = -\left(-\frac{1}{h^2}\right)u_0 + \left(-\frac{2}{h^2}\right)u_2 + \left(\frac{1}{h^2}\right)u_1 \Rightarrow \begin{cases} m_0 = -\frac{1}{h^2}, \\ m_2 = -\frac{2}{h^2}, \\ m_1 = \frac{1}{h^2}. \end{cases} \quad (103)$$

Case (b):

$$\begin{pmatrix} \frac{2}{h^4} & 0 \\ 0 & \frac{1}{2h^2} \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial^2 u_0}{\partial x^2} \end{pmatrix} = \begin{pmatrix} -\frac{u_2}{h^5} + \frac{u_3}{h^5} \\ -\frac{1}{h^4}u_0 + \frac{u_3}{2h^4} + \frac{u_2}{2h^4} \end{pmatrix} \quad (104)$$

then the star equation (21) is

$$\mathfrak{I}[u] = -\left(\frac{2}{h^2}\right)u_0 + \left(\frac{1}{h^2}\right)u_2 + \left(\frac{1}{h^2}\right)u_3 \Rightarrow \begin{cases} m_0 = \frac{2}{h^2}, \\ m_2 = \frac{1}{h^2}, \\ m_3 = \frac{1}{h^2}. \end{cases} \quad (105)$$

Appendix C

C.1. Truncation error for 2D case ($p = 5$)

The extended function for the 2D case $B_S^*[u]$, is given by

$$B_S^*[u] = \sum_{i=1}^N \left[\begin{pmatrix} u_0 - u_i + h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} \\ + \frac{k_i^2}{2} \frac{\partial^2 u_0}{\partial y^2} + h_i k_i \frac{\partial^2 u_0}{\partial x \partial y} + \frac{h_i^3}{6} \frac{\partial^3 u_0}{\partial x^3} \\ + \frac{h_i^2 k_i}{2} \frac{\partial^3 u_0}{\partial x^2 \partial y} + \frac{k_i^2 h_i}{2} \frac{\partial^3 u_0}{\partial y^2 \partial x} + \frac{k_i^3}{6} \frac{\partial^3 u_0}{\partial y^3} \\ + \frac{h_i^4}{24} \frac{\partial^4 u_0}{\partial x^4} + \dots \end{pmatrix} w(h_i, k_i) \right]^2 \quad (106)$$

then

$$\text{TE}_{p=5} = (\gamma_1, \dots, \gamma_5) A_5^{-1} \begin{pmatrix} -\sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^4 + \dots \right] w^2 . h_i \\ -\sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^4 + \dots \right] w^2 . k_i \\ -\sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^4 + \dots \right] w^2 . \frac{h_i^2}{2} \\ -\sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^4 + \dots \right] w^2 . \frac{k_i^2}{2} \\ -\sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} \right)^4 + \dots \right] w^2 . h_i k_i \end{pmatrix} \quad (107)$$

taking into account (108), which can be obtained in a similar way that the one used in 1D case to obtain (44)

$$m_i = \frac{\mu_i}{h_i^2 + k_i^2}, \quad \mu_i \in \mathbb{R} \quad (108)$$

and operating, we obtain

$$\begin{aligned} \text{TE}_{p=5} = & -\sum_{i=1}^N \mu_i \left(\frac{\partial^3 u_0}{\partial x^3} \frac{h_i^3}{6(h_i^2 + k_i^2)} + \frac{\partial^3 u_0}{\partial x^2 \partial y} \frac{h_i^2 k_i}{2(h_i^2 + k_i^2)} \right. \\ & \left. + \frac{\partial^3 u_0}{\partial x \partial y^2} \frac{h_i k_i^2}{2(h_i^2 + k_i^2)} + \frac{\partial^3 u_0}{\partial y^3} \frac{k_i^3}{6(h_i^2 + k_i^2)} \right) \\ & -\sum_{i=1}^N \mu_i \left(\frac{\partial^4 u_0}{\partial x^4} \frac{h_i^4}{24(h_i^2 + k_i^2)} + \frac{\partial^4 u_0}{\partial x^3 \partial y} \frac{h_i^3 k_i}{6(h_i^2 + k_i^2)} + \frac{\partial^4 u_0}{\partial x^2 \partial y^2} \frac{h_i^2 k_i^2}{4(h_i^2 + k_i^2)} \right. \\ & \left. + \frac{\partial^4 u_0}{\partial x \partial y^3} \frac{h_i k_i^3}{6(h_i^2 + k_i^2)} + \frac{\partial^4 u_0}{\partial y^4} \frac{k_i^4}{24(h_i^2 + k_i^2)} \right) + O(h_i^3, k_i^3). \end{aligned} \quad (109)$$

The total truncation errors for 2D case for parabolic and hyperbolic equations are given by

(a) Parabolic equation:

$$\text{TTE}(\text{parabolic}) = -\frac{\Delta t}{2} \frac{\partial^2 u(\{x_0\}, t_1)}{\partial t^2} + \text{TE}_{p=5} + O((\Delta t)^2). \quad (110)$$

(b) Hyperbolic equation:

$$\text{TTE}(\text{hyperb}) = -\frac{(\Delta t)^2}{12} \frac{\partial^4 u(\{x_0\}, t_1)}{\partial t^4} + \text{TE}_{p=5} + O((\Delta t)^4) \quad (111)$$

C.2. Truncation error for 3D case ($p = 9$)

The extended function for the 3D case $B_9^*[u]$, is given by

$$B_9^*[u] = \sum_{i=1}^N \left[\left(\begin{aligned} &u_0 - u_i + h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \\ &+ \frac{h_i^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{k_i^2}{2} \frac{\partial^2 u_0}{\partial y^2} + \frac{p_i^2}{2} \frac{\partial^2 u_0}{\partial z^2} \\ &+ h_i k_i \frac{\partial^2 u_0}{\partial x \partial y} + h_i p_i \frac{\partial^2 u_0}{\partial x \partial z} + p_i k_i \frac{\partial^2 u_0}{\partial z \partial y} \\ &+ \frac{h_i^3}{6} \frac{\partial^3 u_0}{\partial x^3} + \frac{k_i^3}{6} \frac{\partial^3 u_0}{\partial y^3} + \frac{p_i^3}{6} \frac{\partial^3 u_0}{\partial z^3} + \dots \end{aligned} \right) w(h_i, k_i, p_i) \right]^2 \quad (112)$$

then

$$\text{TE}_{p=9} = (\gamma_1, \dots, \gamma_9) A_9^{-1} \left(\begin{aligned} & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 h_i \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 k_i \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 p_i \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 \frac{h_i^2}{2} \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 \frac{k_i^2}{2} \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 \frac{p_i^2}{2} \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 h_i k_i \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 h_i p_i \\ & - \sum_{i=1}^N \left[\frac{1}{3!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^3 + \frac{1}{4!} \left(h_i \frac{\partial u_0}{\partial x} + k_i \frac{\partial u_0}{\partial y} + p_i \frac{\partial u_0}{\partial z} \right)^4 + \dots \right] w^2 k_i p_i \end{aligned} \right) \quad (113)$$

taking into account (114), which can be obtained in a similar way that the one used in 1D case to obtain (44)

$$m_i = \frac{\mu_i}{h_i^2 + k_i^2 + p_i^2}, \quad \mu_i \in \mathbb{R} \quad (114)$$

and operating, we obtain

$$\begin{aligned}
 \text{TE}_{p=9} = & - \sum_{i=1}^N \mu_i \left(\begin{aligned} & \frac{\partial^3 u_0}{\partial x^3} \frac{h_i^3}{6(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^3 u_0}{\partial x^2 \partial y} \frac{h_i^2 k_i}{2(h_i^2 + k_i^2 + p_i^2)} \\ & + \frac{\partial^3 u_0}{\partial x^2 \partial z} \frac{h_i^2 p_i}{2(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^3 u_0}{\partial y^3} \frac{k_i^3}{6(h_i^2 + k_i^2 + p_i^2)} \\ & + \frac{\partial^3 u_0}{\partial x \partial y^2} \frac{h_i k_i^2}{2(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^3 u_0}{\partial y^2 \partial z} \frac{h_i^2 p_i}{2(h_i^2 + k_i^2 + p_i^2)} \\ & + \frac{\partial^3 u_0}{\partial z^3} \frac{p_i^3}{6(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^3 u_0}{\partial x \partial z^2} \frac{h_i p_i^2}{2(h_i^2 + k_i^2 + p_i^2)} \\ & + \frac{\partial^3 u_0}{\partial y \partial z^2} \frac{k_i p_i^2}{2(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^3 u_0}{\partial x \partial y \partial z} \frac{h_i k_i p_i}{(h_i^2 + k_i^2 + p_i^2)} \end{aligned} \right) \\
 & - \sum_{i=1}^N \mu_i \left(\begin{aligned} & \frac{\partial^4 u_0}{\partial x^4} \frac{h_i^4}{24(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^4 u_0}{\partial x^3 \partial y} \frac{h_i^3 k_i}{6(h_i^2 + k_i^2 + p_i^2)} \\ & + \frac{\partial^4 u_0}{\partial x^2 \partial y^2} \frac{h_i^2 k_i^2}{4(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^4 u_0}{\partial x^3 \partial z} \frac{h_i^3 p_i}{6(h_i^2 + k_i^2 + p_i^2)} \\ & + \frac{\partial^4 u_0}{\partial x^2 \partial z^2} \frac{h_i^2 p_i^2}{4(h_i^2 + k_i^2 + p_i^2)} + \dots \\ & + \frac{\partial^4 u_0}{\partial y^4} \frac{k_i^4}{24(h_i^2 + k_i^2 + p_i^2)} + \frac{\partial^4 u_0}{\partial z^4} \frac{p_i^4}{24(h_i^2 + k_i^2 + p_i^2)} \\ & + \dots \end{aligned} \right) + O(h_i^3). \quad (115)
 \end{aligned}$$

The total truncation errors for 3D case for parabolic and hyperbolic equations are given by

(a) Parabolic equation:

$$\text{TTE}(\text{parabolic}) = -\frac{\Delta t}{2} \frac{\partial^2 u(\{x_0\}, t_1)}{\partial t^2} + \text{TE}_{p=9} + O((\Delta t)^2). \quad (116)$$

(b) Hyperbolic equation:

$$\text{TTE}(\text{hyperb}) = -\frac{(\Delta t)^2}{12} \frac{\partial^4 u(\{x_0\}, t_1)}{\partial t^4} + \text{TE}_{p=9} + O((\Delta t)^4). \quad (117)$$

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